

BENDING MOMENT INTERPRETATION OF STRUCTURAL ELEMENT WITH MEASURED DEFLECTION PROFILE

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ABSTRACT

There are many measured deflection profiles in the flexural structural elements, like piles, walls and etc, obtained from the instrumentation programme of the projects. However, the use of these measurements is only limited to comparison the measured deflection against the predetermined threshold values in the design criteria. It could be due to the tedious process involved that further interpretation on the induced moment is not being seriously carried out to get the best value out of the verification processes. It is certainly very wasteful for not doing so. As such, this paper presents a procedure for back-calculation of bending moment based on the valuable measured deflection profile. Consideration is also given to the post-cracking behaviour of a reinforced concrete (RC) flexural element and the computation of bending moment of the cracked section in relation to the measured curvature. A case history of using the suggested procedures to successfully compute the bending moment of RC piled raft foundation is presented.

Keywords: Deflection profile; Flexural elements; Instrumentation; Back-calculation; Bending moment; Curvature; Crack section.

1. INTRODUCTION

This paper presents the procedures for determination of bending moment based on best-fit curve method on measured deflection profile of a R.C. flexural element. Basically, the overall procedure consists of three major steps, which are; (a) determination for curvature, k , (b) determination for effective moment of inertia, I_e and (c) determination for bending moment, M . The above steps are described and explained in details including the concepts involved.

In the following sections, methods for determining the equation of the deflection curve and curvature at specific points along the axis of the beam are described. At the later part in this paper, examples and results by using the above procedure to determine bending moment based on the measured data at site is presented.

There is another way to assess the bending moment in a flexural structural element by measuring the strain in both the compressive and tensile zone of a concrete section. By applying a concrete stress-strain model, one can derive the bending moment in a section for equilibrium of stress across whole concrete section. Ng, et al (1992) have presented a case history on determination of bending moment in a concrete diaphragm wall and concluded that the concrete model proposed by Scott (1983) can offer the best prediction of behavior. However, this method is not covered under the scope of this paper.

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2. THEORY OF BEAM DEFLECTION

Gere & Timoshenko (1984) have given a very detailed derivation of the beam curvature using the first and second order derivatives of a deflection curve of a beam. Figure 1a shows the details of the beam deflection.

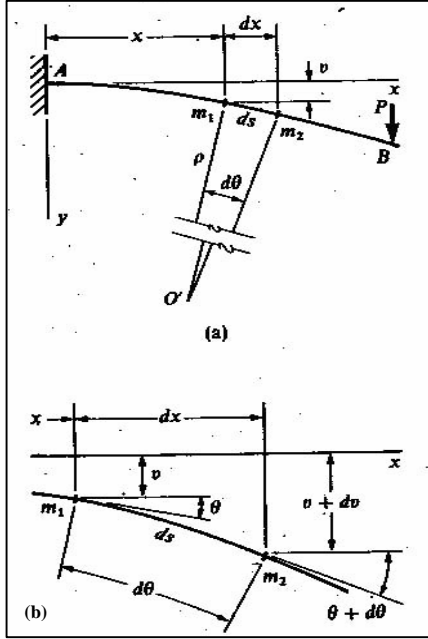


Figure 1 : Deflection Curve of Beam

If the origin of coordinates is assumed at the fixed end, with the x -axis directed to the right and the y -axis directed downward. The xy plane is a plane of symmetry and all loads act in this plane; thus, the xy plane is the plane of bending. The deflection v of the beam at any point m_1 at distance x from the origin is the translation (or displacement) of that point in the y direction, measured from the x -axis to the deflection curve.

For the selected axes, a downward deflection is positive and upward deflection is negative, the equation of the deflection curve can be expressed as a function of x .

The angle of rotation q of the axis of the beam at any point m_1 is the angle between the x -axis and the tangent to the deflection curve (Figure 1(b)). This angle is positive when clockwise, provided the x and y axes have the directions as shown in Figure 1(a).

Now considering a second point m_2 , located on the deflection curve at a small distance ds further along the curve and at distance $x + dx$ (measured parallel to the x axis) from the origin, the deflection at this point is $v + dv$, where dv is the increment in deflection as we move from m_1 to m_2 . Also, the angle of rotation at m_2 is $q + dq$, where

dq is the increment in angle of rotation. The intersection of two lines normal to the tangents at points m_1 and m_2 locates the center of curvature O' and the distance from O' to the curve is the radius of curvature r . From the figure, it can be denoted that $rdq = ds$; hence, the curvature k (reciprocal of the radius of curvature) is given by the following equation:

$$k = \frac{1}{r} = \frac{dq}{ds} \quad (1)$$

The slope of the deflection curve is the first derivative dv/dx and is equal to the tangent of the angle of rotation q . Due to the reason that dx is infinitesimally small; thus

$$\frac{dv}{dx} = \tan q \quad \text{or} \quad q = \arctan \frac{dv}{dx} = \arctan v' \quad (2)$$

From Eq-1 and Eq-2, the expression of curvature can be further derived as follows;

$$k = \frac{1}{r} = \frac{dq}{ds} = \frac{d(\arctan v')}{dx} \frac{dx}{ds} \quad (3)$$

From Figure 1b where $ds^2 = dx^2 + dv^2$, the following expression can be obtained;

$$\frac{ds}{dx} = \left[1 + \left(\frac{dv}{dx} \right)^2 \right]^{\frac{1}{2}} = \left[1 + (v')^2 \right]^{\frac{1}{2}} \quad (4)$$

By differentiation,
$$\frac{d(\arctan v')}{dx} = \frac{v''}{1+(v')^2} \quad (5)$$

Substitution of these expressions (Eq-4 and Eq-5) into the equation for curvature (Eq-3) yields

$$k = \frac{1}{r} = \frac{v''}{[1+(v')^2]^{\frac{3}{2}}} \quad (6)$$

It is obvious that the assumption of small slopes is equivalent to disregarding $(v')^2$, thus making the denominator in Eq-6 equal to one; where v' and v'' denotes dy/dx and d^2y/dx^2 respectively.

Sometimes, the following simplified expression is used for computing curvature of beam undergoing only very small rotation when loaded.

$$k = \frac{1}{r} = v'' \quad (7)$$

This can lead to serious error if the rotation of the deflection beam at the point of interest is large. The following example illustrates the erratic results. Figure 2 shows the function of a circle; the derivations of the curvature are as follows:

Table 1 Computation of Curvature on Circle

Point	x/R	y/R	$v' = \frac{dy}{dx}$	$v'' = \frac{d^2y}{dx^2}$	$k = \frac{v''}{[1+(v')^2]^{\frac{3}{2}}}$
A	0	1.000	0	-1	-1/R
B	0.707	0.707	-1	-2.828	-1/R
C	1.000	0	∞	∞	-1/R *

* refer to derivation below

Circular equation : $y = \sqrt{R^2 - x^2}$

First derivation : $v' = \frac{dy}{dx} = -\frac{x}{\sqrt{R^2 - x^2}}$

Second derivation : $v'' = \frac{d^2y}{dx^2} = -\frac{R^2}{(R^2 - x^2)^{\frac{3}{2}}}$

$$v' = \frac{dy}{dx} = -\frac{x}{\sqrt{R^2 - x^2}} \Rightarrow (v')^2 = \frac{x^2}{R^2 - x^2}$$

$$v'' = -\frac{R^2}{(R^2 - x^2)^{\frac{3}{2}}}$$

$$k = \frac{1}{r} = \frac{v''}{[1+(v')^2]^{\frac{3}{2}}} = \frac{-R^2}{(R^2 - x^2)^{\frac{3}{2}} \left[1 + \left(\frac{x^2}{R^2 - x^2}\right)\right]^{\frac{3}{2}}} = \frac{-R^2}{(R^2 - x^2)^{\frac{3}{2}} \left[\frac{R^2 - x^2 + x^2}{(R^2 - x^2)}\right]^{\frac{3}{2}}} = \frac{-R^2}{(R^2 - x^2)^{\frac{3}{2}} \frac{R^3}{(R^2 - x^2)^{\frac{3}{2}}}} = -1/R$$

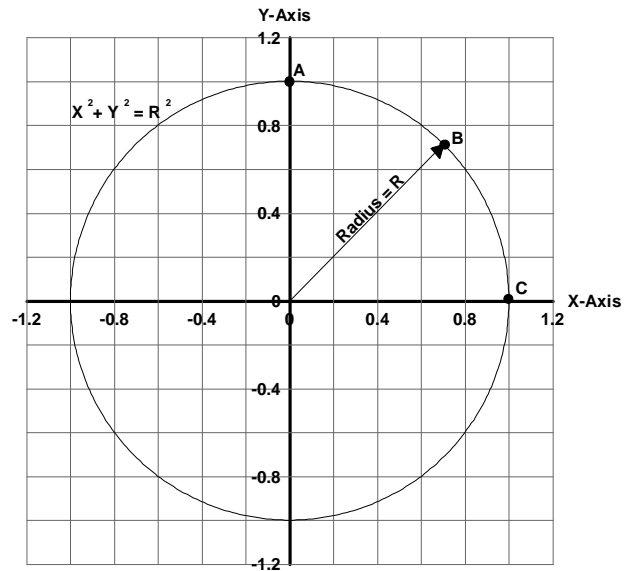


Figure 2 : Function of a Circle

Table 1 shows the comparison between the simplified expression (Eq-7) and the exact expression (Eq-6) for curvature. It is clear that the exact expression shows the same value at all points. However, the simplified expression give a very erroneous results at points B and C of a circle except at point A where the tangential gradient is parallel to the axis ($v' = 0$). Therefore, it advisable to use the exact expression for computation of curvature.

3. PROCEDURE FOR DETERMINATION OF CURVATURE

This section describes the steps in determining curvature (κ) of a measured deflection curve. First, The measured deflection profile (with all the data points) in the flexural structural elements, like piles, wall and others is obtained by curve fitting on the measured displacement of the individual surveyed points. Best-fit polynomial curve for a cluster of neighboring data points are obtained and the curvature can be computed at the middle point of the neighboring data points by using differentiation of polynomial equation. In this paper, a sixth degree polynomial equation is proposed with the use of Microsoft Excel spreadsheet.

The first and second derivatives on the above best-fit six-degree polynomial at the 4th point yield the value of v' and v'' respectively. Substituting these values in Eq-6 yield the value of curvature at the point of consideration.

The above steps are repeated for the next 7 data points starting from 2nd to 8th data point and so on until the curvature of all the respective data points along the measured deflection profile are determined.

Currently, most of the available software is capable to fit the data cluster with a polynomial of not more than 10 degrees. Due to such limitation, all the measured data points should not be best-fitted at one go but rather to fit them in several clusters of seven data with six degree polynomial as this is the most practical solution and optimised.

As suggested by Poh et al.(1999), the use of a lower degree curve fitting function, for example a third- to fifth-degree polynomial, may tend to flatten out some of the local peaks in the curve and hence may underestimate the corresponding curvature. Therefore, a minimum sixth-degree polynomial function is suggested in order to capture the localized larger curvature.

4. PROCEDURE FOR DETERMINATION OF MOMENT OF INERTIA

For any reinforced concrete section, the moment of inertia varies with the extent of cracking at the section. When the applied bending moment (M) is sufficiently small where the corresponding outer fibre concrete tensile stress does not exceed the tensile strength or the modulus rupture of concrete, the section will remain uncracked. Under these circumstances, the moment of inertia of the uncracked section can be assumed to be the gross moment of inertia (I_g) of a transformed section.

For a crack section ($M \geq M_{cr}$, where M_{cr} = cracking moment of the concrete section), the moment of inertia of the section is less than the gross moment of inertia and varies with the extent of cracking at the section. However, for the intact concrete block between the cracks, the concrete in tension continues to contribute to the flexural stiffness and therefore reduces the curvature. This phenomenon is called tension stiffening (Ghali 1993). Because of the tension stiffening effect, the actual curvature value at a particular section (partially cracked section) of the intact concrete block between the cracks is smaller than the curvature value for a fully cracked section (with the concrete in tension ignored except the transformed area for tension steels still continues to provide tension) but is larger than the curvature value for an uncracked section. Similarly, the moment of inertia for a partially cracked section is less than the gross moment of inertia of the uncracked concrete section but is larger than the moment of inertia for a completely cracked section with the concrete in tension ignored except the tension in tension steel. In this case, an effective moment of inertia (I_e) should be used in the analysis of a cracked section.

For a rectangular flexural concrete member without axial loading, the value of I_e as proposed by Branson (1977) as follows can be used to calculate the effective moment of inertia:

$$I_e = \left[\frac{M_{cr}}{M} \right]^4 I_g + \left[1 - \left(\frac{M_{cr}}{M} \right)^4 \right] I_{cr} \leq I_g \quad (8)$$

where

I_{cr} = moment of inertia of a transformed cracked section.

M_{cr} = cracking moment for the section = $\frac{f_r I_g}{y_t}$

f_r = modulus of rupture of concrete

= $0.623\sqrt{f_c'}$ (in MPa)

f_c' = cylinder compressive strength of concrete (in MPa)

= $(f_{cu}) / 1.25$

f_{cu} = cube compression strength of concrete (in MPa)

y_t = the distance from the centroidal axis of cross section, neglecting steel, to the extreme fibre tension.

Kong & Evans (1987) show that the moment of inertia of a cracked section for a rectangular flexural member can be calculated from the following equation:

$$\frac{I_{cr}}{bd^3} = \frac{1}{3}\left(\frac{x}{d}\right)^3 + a_e r \left(1 - \frac{x}{d}\right)^2 + a_e r' \left(\frac{x}{d} - \frac{d'}{d}\right)^2$$

in which

$$\frac{x}{d} = -a_e (r + r') + \sqrt{a_e^2 (r + r')^2 + 2a_e (r + \frac{d'}{d} r')}$$

where

d' = effective depth for the reinforced concrete section

d = depth from the compression face to the centroid of compression steel.

x = neutral axis

a_e = modulus ratio (E_s/E_c)

r' = compression steel ratio = A_s'/bd

r = tension steel ratio = A_s/bd

A_s = tension steel area

A_s' = compression steel

b = width of the reinforced concrete section

Obviously the effective moment of inertia of a given RC section is dependent of the applied moment to the section.

5. DETERMINATION FOR BENDING MOMENT FROM MEASURED CURVATURE

The bending moment equation can be expressed as follow:

$$M = \kappa EI_e \tag{9}$$

where

M = bending moment on the RC section

κ = measured curvature of RC section

E = Young modulus of concrete

I_e = effective moment of inertia

The derivation of Eq-9 shall be explained in detailed as follows; Figure 3 shows the deformation of a beam in pure bending produced by couples M_0 . (note: the bending moment M equals $-M_0$)

Under the action of the couples, M_0 , the beam deflects in the xy plane and its axis is bent into a circular curve (Figure 3c). Cross sections of the beam, such as mn and pq , remain plane and normal to longitudinal lines, or fibers, of the beam. The symmetry of the beam and its loading (Figure 3a) requires that all elements of the beam (such as element $mnpq$) deform in an identical manner, which is possible only if the deflection curve is circular (Figure 3c) and if cross sections remain plane during loading.

As a result of the bending deformations, cross sections mn and pq rotate with respect to each other about axes perpendicular to the xy plane. The longitudinal fibers on the convex side of the beam are elongated, whereas those on the concave side are shortened. Thus, the fibers in the upper part of the beam are in tension, whereas those in the lower part are in compression. Somewhere between the top and bottom of the beam is a surface in which the longitudinal fibers do not change in length. This surface, indicated by the dashed line ss in Figures 3a and 3c, is called the neutral surface of the beam.

The plane of cross sections mn and pq of the deformed beam intersect in a line through the center of curvature O' (Figure 3c). The angle between these planes is denoted $d\theta$, and the distance from O' to the neutral surface is the radius of curvature ρ . The initial distance dx between the two planes is unchanged at the neutral surface, hence $\rho d\theta = dx$. However, all other longitudinal fibers either lengthen or shorten, thereby creating longitudinal ϵ_x . Let us consider a typical longitudinal fiber ef located within the beam at a distance y from the neutral surface (Figure 3c). The length L_1 of this fiber is

$$L_1 = (r - y)d\theta = dx - \frac{y}{r} dx$$

In as much as the original length of ef is dx , it follows that its elongation is $L_1 - dx$, or $-ydx/r$. The corresponding strain is equal to the elongation divided by the initial length dx ; hence:

$$\epsilon_x = -\frac{y}{r} = -ky \tag{10}$$

where $\kappa = \text{curvature}$

If the material is elastic with a linear stress-strain diagram, we can use Hooke's law for uniaxial stress ($\sigma = E\epsilon$) and obtain

$$s_x = E\epsilon_x = -Eky \tag{11}$$

Thus, the normal stresses acting on the cross section vary linearly with distance y from the neutral surface. This type of stress distribution is pictured in Figure 4a, where the stresses are negative (compression) below the neutral surface and positive (tension) above the surface when the applied couple, M_0 , acts in the direction shown.

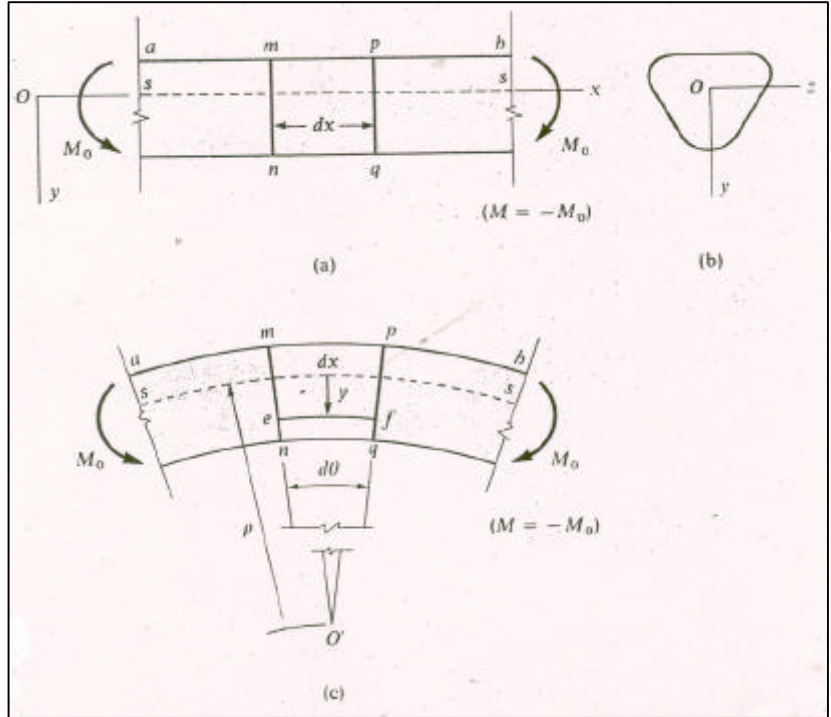


Figure 3 : Deformation of a Beam in Pure Bending Moment Produced by Couples M_0

The resultant of the normal stresses σ_x acting over the cross section consists of a horizontal force in the x direction and a couple acting about the z axis. An element of area dA is considered in the cross section with a distance y from the neutral axis (Figure 4b). The force acting on this element is normal to the cross section and has a magnitude $\sigma_x dA$. Because no resultant normal force acts on the cross section, the integral of $\sigma_x dA$ over the entire area of the cross section must vanish; thus,

$$\int \mathbf{s}_x dA = -\int Eky dA = 0$$

Because the curvature κ and modulus of elasticity E are constants at the cross section, therefore it can be concluded that

$$\int y dA = 0$$

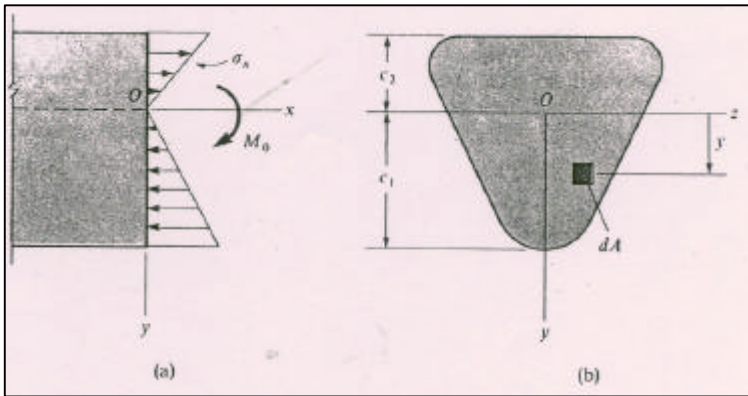


Figure 4 : Distribution of Normal Stress σ_x in a Beam of Linear Elastic Material

for a beam in pure bending and hence, the neutral axis passes through the centroid of the cross section when the material of the beams follows Hooke's law. Therefore, when a beam of linear elastic material is subjected to pure bending, the y and z axes are principal centroidal axes.

When the moment resultant of the stresses σ_x is considered acting over the cross section (Figure 4a), its moment about the z axis, which represents the infinitesimal contribution of $\sigma_x dA$ to the moment, M_o , is

$$dM_o = -\mathbf{s}_x y dA$$

The integral of all such elemental moments over the entire cross-sectional area must result in the total moment M_o ; thus,

$$M_o = -\int \mathbf{s}_x y dA$$

Noting again that the bending moment M is equal to $-M_o$, and also substituting for σ_x from Eq-11, the following expression can be derived:

$$M = \int \mathbf{s}_x y dA = -kE \int y^2 dA$$

This equation can be expressed in the following simpler form

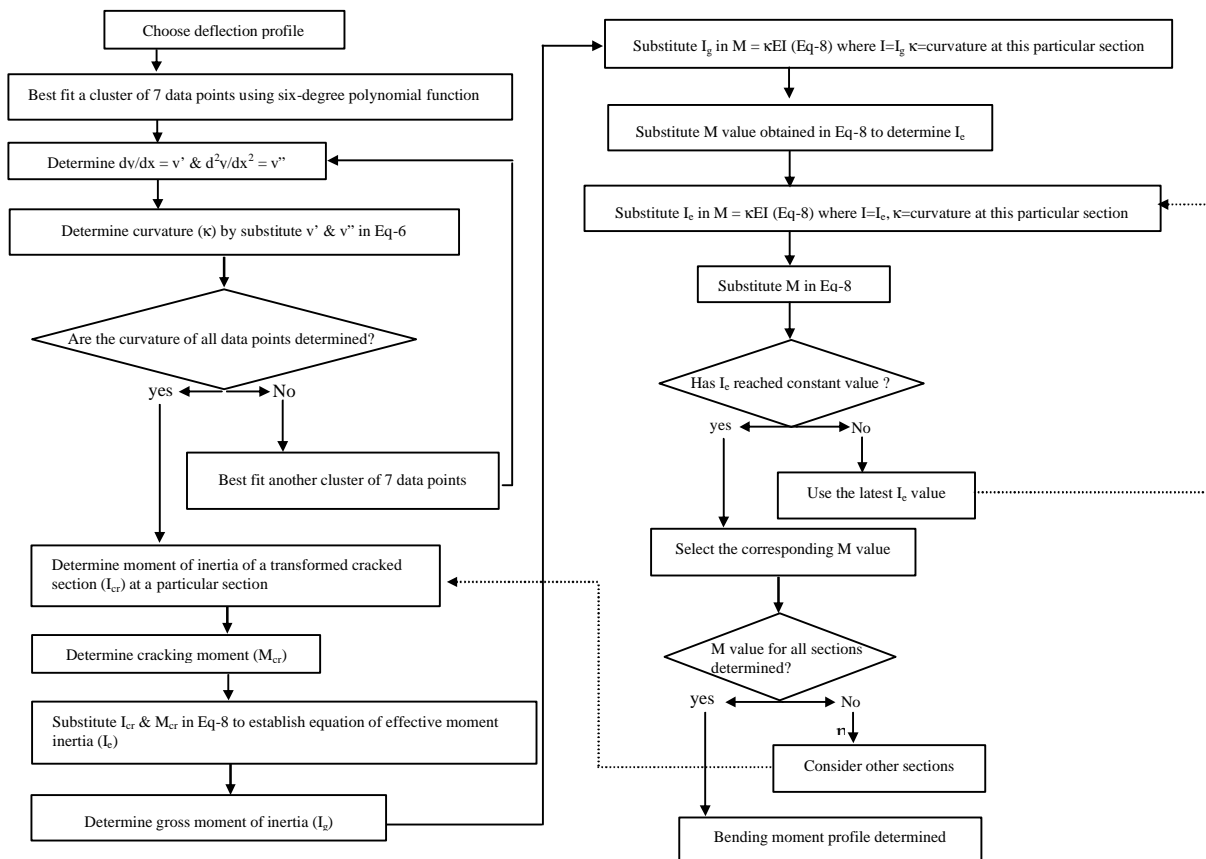
$$M = -kEI$$

in which $I = \int y^2 dA$

The minus sign in the above equation is a consequence of the sign convention that has been adopted for bending moment. If the opposite sign convention for bending moments is used, or if the y axis is positive upward, then the minus sign is omitted.

As both M and I_e are mutually dependent, iterative process is required to solve Eq-8 and Eq-9. Starting from gross moment of inertia (I_g), this value is then substituted in Eq-9 (where $I_g = I_e$). The M value obtained is then substituted in Eq-8. The I_e value is substituted back in Eq-9. The iteration steps are repeated until both M and I_e show constant values. The above steps are repeated for other points along the measured deflection profile until the whole bending moment profile are determined.

6. FLOW CHART FOR DETERMINATION OF BENDING MOMENT



7. CASE HISTORY

An example of a case history is presented here by using the above-mentioned method to calculate bending moment based on measured deflection profile. The project was a palm oil mill constructed on a sand filled platform of about 83,000m² overlying a soft swampy ground at the distance of about 50km away from the river mouth of Sg. Guntung at Riau, Indonesia. This project involved construction of seven R.C piled raft foundation supporting 2500tons oil/water storage steel tank. Liew et al. (2002) has presented this case history.

An 85mm diameter horizontal inclinometer access tubing was installed at the middle of the 500mm thick 19.5m diameter R.C raft foundation with which supports a 18m diameter steel tank on top. The purpose of installing inclinometer access tubing is to facilitate

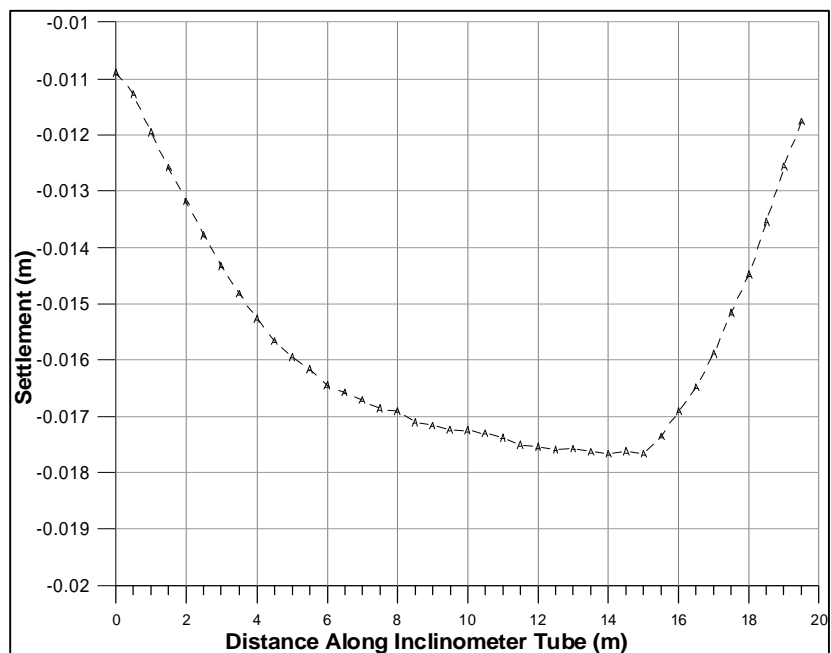


Figure 5 : Deflection Profile on RC Raft

inclinometer probe to traverse from one end to another end across the raft. Horizontal inclinometer was used to obtain the vertical deflection profile of R.C raft. During water testing, the R.C. raft will deflect according to the total imposed water load on the raft. The horizontal sensor was drawn through the access tube from one end to another and reading was taken at 0.5m intervals. The probe was then reversed end-for-end and pulled for second time through the tube at the similar interval for systematic correction of drifting.

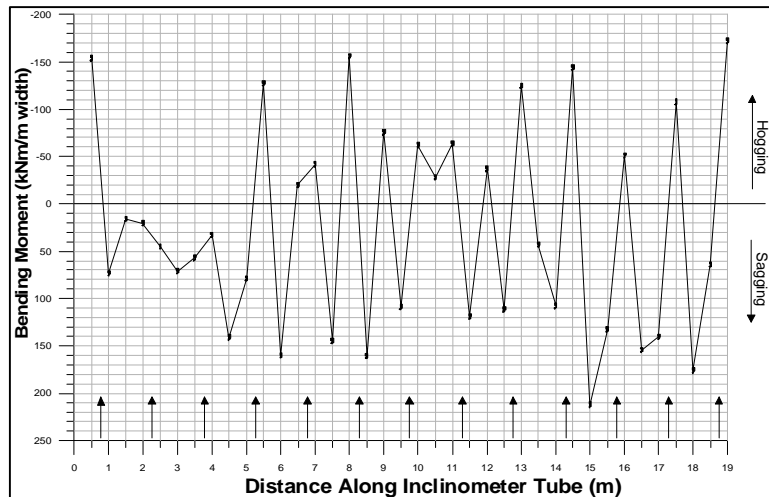


Figure 6 : Bending Moment Profile on R.C. Raft Foundation

At any increment or decrement of loading (by increase or decrease water level) on the raft, the inclinometer readings were taken in order to determine the deflection profile with the respective loading stage. The reduced levels of 2 markers located at both end of the access tubing were then used to calibrate the inclinometer readings accordingly to check and produce the overall deflection profile.

Figure 5 shows the horizontal deflection profile on R.C raft foundation during water testing at 2500tons (maximum serviceability load). Based on the deflection profile obtained, the bending moment profile of the raft in accordance to the maximum serviceability loading can be calculated with the procedure as discussed in the earlier sections.

Figure 6 shows the bending moment profile on R.C raft foundation derived from the measured horizontal deflection profile. From the figure, the bending moment fluctuate along the center alignment of the raft. The fluctuation of the interpreted bending moment reflects the localized hogging moment at the pile support and the sagging moment at the mid point between the pile supports. The arrows pointed upward denotes the pile position supported underneath the raft foundation.

8. DISCUSSION

This proposed method has the following advantages in the interpretation of bending moment:

- a) It provides continues bending moment profile through intrapolation of the discrete instrumentation data.
- b) The interpreted bending moment profile can provide a good reveal of the moment distribution for confirming the safety of the structural element and for further value engineering where possible.

However, there are also limitations of the proposed method as follows:

- c) The accuracy of the curvature will reduce when approaches to the both ends of the deflection profile. This is because there are no data beyond the starting point or ending point for a balanced polynomial curve fitting. It is advisable to increase the extent of monitored deflection profile to adequately cover the zone of interest.
- d) The accuracy on the determination of the modulus of concrete will affect the interpreted bending moment as the modulus of concrete is governed by the strength, age, stress level and etc. To overcome this problem, compression test can be carried out on the cored concrete samples to determine the actual concrete modulus for the duration of monitoring.

9. CONCLUSIONS & RECOMMENDATIONS

From the discussion in the earlier sections, the following conclusion and recommendations can be made:

- (a) This paper presents the procedural steps in determining the bending moment profile of flexural structural elements based on best-fit curve method on measured deflection profiles from instrumentation programme.
- (b) The overall procedures consist of five major steps, which are :
 - Best-fit curve to the measured deflection profile
 - Determination of curvature (k) by differentiation of the best-fit curve
 - Determination of effective moment inertia (I_e)
 - Determination of bending moment (M)
 - Iterative process between I_e and M
- (c) An example of case history is presented to illustrate the successful implementation of the abovementioned method.
- (d) For the future research, it is useful to compare the bending moments interpreted from the deflection profile and also the method using concrete model as proposed by Scott (1983)

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